INFERENCE IN SECOND PRICE AUCTIONS WITH GAMMA DISTRIBUTED COMMON VALUES

BERTIL WEGMANN

Abstract. Our paper explores possible limitations of the Gaussian model in Wegmann and Villani (2008, WV) due to intrinsically non-negative values. The relative performance of the Gaussian model is compared to an extension of the Gamma model in Gordy (2008) within the symmetric second price common value model. A key feature in our approach is the derivation of an accurate approximation of the bid function for the Gamma model, which can be inverted and differentiated analytically. This is extremely valuable for fast and numerically stable evaluations of the likelihood function. The general MCMC algorithm in WV is utilized to estimate WV's eBay dataset from 1000 auctions of U.S. proof coin sets, as well as simulated datasets from the Gamma model with different degrees of skewness in the value distribution. The Gaussian model fits the data slightly better than the Gamma model for the particular eBay dataset, which can be explained by the fairly symmetrical value distribution. The superiority of the Gamma to the Gaussian model is shown to increase for higher degrees of skewness in the simulated datasets.
Keywords:

1. Introduction

Structural econometric models of auction data have become increasingly common in the literature, see e.g. Bajari (2005) and Paarsch and Hong (2006) for recent surveys. As emphasized by Laffont and Vuong (1996), auction models are particularly well suited for structural estimation since many datasets are available and well-defined game forms exists. This is especially true in the field of Internet auctions, such as eBay, where high quality datasets are readibly available.

Most of the work in econometric models of auction data analyzes either the private or the common value model. Within the private value paradigm each bidder knows his own valuation of the object and will not be affected by knowing the valuations of the other bidders. In a common value auction the value of the object is unknown but the same for all bidders, and each bidder uses his private information (the signal) to estimate the unknown value. Good examples of structural econometric auction models are Paarsch (1992), Elyakime, Laffont, Loisel and Vuong (1997), and Bajari and Hortacsu (2003, BH).

Within each paradigm the values of the objects are often assumed to follow a certain distribution that is commonly known to all bidders. In most empirical work the outcomes from one or several value distributions are treated in isolation without comparing the performance between the different distributions. Our paper assume the common value model, and show how the relative performance between two different distributional setups differ substantially for various types of data.

BH model Internet coin auctions at eBay as independent second price common value auctions with stochastic entry, and assume symmetric bidders and a symmetric Nash equilibrium. The objects' values are assumed to follow a hierarchical Gaussian model where the mean, variance, and the expected number of bidders are functions of covariates. Wegmann and Villani

Department of Statistics, Stockholm University, SE-106 91 Stockholm. E-mail: bertil.wegmann@stat.su.se.

(2008, WV) refine and extend the analysis in BH. WV derive an accurate linear approximation of the bid function that can be inverted and differentiated analytically. This is extremely valuable for fast and numerically stable evaluations of the likelihood function. Moreover, WV use a general Metropolis-Hastings algorithm for Bayesian variable selection to quantify the importance of individual covariates in the model. The model appears to fit the data well, and the out-of-sample predictive performance is good.

Since values are intrinsically non-negative, one can argue that the assumption of normally distributed values in BH is untenable. The Gaussian model is nevertheless likely to be a very useful approximation when the coefficient of variation (CV) of the value distribution is moderate or small, so that the distribution has a small probability on negative values. The aim of this paper is to explore this issue by contrasting the Gaussian model with an extension of the Gamma model in Gordy (1998). We use the same dataset as in WV from 1000 eBay auctions of U.S. proof coin sets. Other distributions than Gamma could also be used on $(0,\infty)$, but since the Gamma model is closed under multiplication we are able to derive a suitable and accurate approximate solution of the bid function for the Gamma case. This approximation is non-linear, but of a simple form that can be inverted and differentiated analytically.

The parameter estimates for the two models are nearly the same, in both sign and magnitude, but the Gaussian model performs slightly better in predicting auction prices and the MCMC algorithm is more efficient in the Gaussian case. The reason why the Gaussian distribution does so well is that the inferred signals in this particular eBay dataset are fairly symmetric and essentially bounded away from zero. To explore the limitations of the Gaussian model for other possible datasets, we conducted a simulation study where the data generating process had increasingly more skewness in the signals. As expected, the Gamma model is shown to be superior to the Gaussian model, and the superiority is increasing for higher degrees of skewness.

2. Two hierarchical models

We compare two models of common values in second price auctions with stochastic entry. First, we adopt the same hierarchical Gaussian model as in WV. Second, to account for asymmetric or positive-valued distributions, we extend the Gamma model in Gordy (1998).

2.1. General setup.

Assume the seller sets a publicly announced *minimum bid (public reserve)*, $r \geq 0$, and risk-neutral bidders compete for a single object using the same bidding strategy (symmetric bidders). The value of the object, v, is unknown and the same for each bidder at the time of bidding, but a prior distribution for v is shared by the bidders. To estimate v, each bidder relies upon their own private information of the object to receive a private signal x from the same distribution $x|v$. Let $f_v(v)$ denote the probability density function of v, $f_{x|v}(x|v)$ the conditional probability density function of $x|v$, and $F_{x|v}(x|v)$ the conditional cumulative distribution function of $x|v$. Since the auction involves symmetric bidders and a symmetric equilibrium we can focus on a single bidder without loss of generality. The bid function can be written as[1]

$$b(x,\lambda) = \begin{cases} \dfrac{\sum_{n=2}^{\infty}(n-1)\cdot p_n(\lambda)\cdot\int_{-\infty}^{\infty} v\cdot F_{x|v}^{n-2}(x|v)\cdot f_{x|v}^{2}(x|v)\cdot f_v(v)\ dv}{\sum_{n=2}^{\infty}(n-1)\cdot p_n(\lambda)\cdot\int_{-\infty}^{\infty} F_{x|v}^{n-2}(x|v)\cdot f_{x|v}^{2}(x|v)\cdot f_v(v)\ dv}, \text{ if } x \geq x^{\star} \\ 0, \text{ if } x < x^{\star} \textit{ or } b(x,\lambda) < 0, \end{cases} \tag{2.1}$$

where $p_n(\lambda)$ is the Poisson probability of $(n-1)$ bidders in the auction with λ as the expected value of the Poisson entry process. Bidders participate with a positive bid if their signal,

[1] See BH for an implicit solution.

x, is above the cut-off signal level $x^\star$. Given an arbitrary bidder with signal x, let y be the maximum signal of the other $(n-1)$ bidders. The cut-off signal level is then given in implicit form as (Milgrom and Weber, 1982)

$$x^\star(r,\lambda) = \inf_x \left(E_n E[v|X=x, Y<x, n] \geq r\right),$$

which gives the minimum bid, r, as

$$r(x^\star,\lambda) = \sum_{n=2}^{\infty} p_n(\lambda) \cdot \frac{\int_{-\infty}^{\infty} v \cdot F_{x|v}^{n-1}(x^\star|v) \cdot f_{x|v}(x^\star|v) \cdot f_v(v)\ dv}{\int_{-\infty}^{\infty} F_{x|v}^{n-1}(x^\star|v) \cdot f_{x|v}(x^\star|v) \cdot f_v(v)\ dv}. \tag{2.2}$$

Note that the publicly announced minimum bid, r, is only written as a function of the cut-off signal for tractability. The minimum bid is exogenously given by the seller.

In both the Gaussian and the Gamma models presented below the expected value μ and the variance σ^2 exists in the distribution of v. Moreover, $E[x|v] = v$, and $E\left[\frac{1}{x}|v\right] = \frac{1}{v}$ in the Gaussian and the Gamma model, respectively. Similar to BH we specify regression models for $(\mu_j, \sigma_j^2, \lambda_j)$ in auction j as

$$\begin{aligned} \mu_j &= z'_{\mu_j}\beta_\mu \\ \sigma_j^2 &= \exp\left(z'_{\sigma_j}\beta_\sigma\right) \\ \lambda_j &= \exp\left(z'_{\lambda_j}\beta_\lambda\right), \end{aligned} \tag{2.3}$$

where $z_j = (z'_{\mu_j}, z'_{\sigma_j}, z'_{\lambda_j})'$ are auction specific covariates in auction j.

BH treat strategic considerations of objects in parallel auctions to be independent, and show that last-minute bidding is a symmetric Nash equilibrium on eBay. This allows BH to model their eBay auctions as independent second price auctions. The likelihood function of bids is complicated since some bids become unobserved. First, some bidders may draw a signal $x < x^\star$, in which case they do not place any bid. Second, the highest bid is usually not observed because of eBay's proxy bidding system, see WV for a detailed description. The bid distribution for a single auction is of the form:

$$f_b\left(b|\beta,r,z,v\right) = f_{x|v}\left[\phi\left(b|\beta,r,z\right)|v,\kappa,\sigma\right]\phi'\left(b|\beta,r,z\right), \tag{2.4}$$

where $f_b\left(b|\beta,r,z,v\right)$ is the probability density function of the bids conditional on (β,r,z,v), and $\phi\left(b|\beta,r,z\right)$ is the inverse bid function given (β,r,z). Let n be the number of bidders who submit a positive bid in a given auction and let $\mathbf{b} = (b_2, b_3, \ldots, b_n)$ be the vector of observed bids. Then, the likelihood function for that auction is given by

$$\begin{aligned} &f_{\mathbf{b}}\left(b_2, b_3, \ldots, b_n|\mu,\sigma,\lambda,\beta,r,z\right) \\ &= \sum_{i=n}^{\bar{N}} p(i|\lambda) \cdot \int_{-\infty}^{\infty} F_{x|v}\left(x^\star|v,\kappa,\sigma\right)^{i-n} \cdot \left\{1 - F_{x|v}\left[\phi\left(b_2|\beta,r,z\right)|v,\kappa,\sigma\right]\right\}^{I(n\geq 1)} \\ &\quad \times \prod_{i=2}^{n} f_b\left(b_i|\beta,r,z,v\right) \cdot f_v(v|\mu,\sigma) dv, \end{aligned} \tag{2.5}$$

where $p(i|\lambda)$ is the Poisson probability of i bidders in the auction with λ as the expected value, $I\left(n \geq 1\right)$ is an indicator variable for at least one bidder in the auction, and $\bar{N}$ is an upper bound for the total number of potential bidders. For the sake of tractability, let $\bar{N} = 30$, as in BH.

To estimate the parameters of the models we combine the likelihood function with the same prior settings and Bayesian techniques as in WV. A single evaluation of the posterior (likelihood) requires numerical integration to compute $b(x|\beta, r, z)$ in (2.1), followed by additional numerical work to invert and differentiate $b(x|\beta, r, z)$. The same applies to the computation of $x^\star$. This costly procedure needs to be repeated for each of the auctions in the dataset. Instead, we make use of bid approximations for both models which leads to much faster and numerically stable likelihood evaluations. We first review the linear approximation of the bid function in WV for the hierarchical Gaussian model. Then, we derive the approximate bid function for the Gamma model below.

2.2. **Gaussian model.** Let v_j denote the common value in auction j, and let x_{ij} denote the signal of the ith bidder in auction j. The hierarchical Gaussian model can be defined as

$$\begin{aligned} v_j &\sim N(\mu_j, \sigma_j^2), \;\; j = 1, ..., m, \\ x_{ij}\,|v_j &\overset{iid}{\sim} N(v_j, \kappa\sigma_j^2), \;\; i = 1, ..., n_j, \end{aligned} \tag{2.6}$$

where m is the total number of auctions, and n_j the number of bidders that bid zero or place a positive bid in auction j. To get much faster and numerically stable likelihood evaluations we use the linear approximation of the bid function in WV,

$$b_{appr}(x, \lambda) = \begin{cases} c + \omega\mu + (1-\omega)x, \text{ if } x \geq x^\star \\ 0, \text{ if } x < x^\star \textit{ or } b(x, \lambda) < 0, \end{cases} \tag{2.7}$$

where $c = -\frac{\sqrt{\kappa}\sigma\gamma\theta(\lambda-2)}{\gamma(\lambda-2)+1+\frac{\kappa}{2}}$, $\omega = \frac{\frac{\kappa}{2}}{\gamma(\lambda-2)+1+\frac{\kappa}{2}}$, $\theta = 1.96$ and $\gamma = 0.1938$. In addition, WV show that the cutoff signal can be similarly approximated by

$$x^\star(r, \lambda) \approx \frac{r - \sum_{n=2}^{\infty} p_n(\lambda)(\tilde{c} + \tilde{\omega}\mu)}{\sum_{n=2}^{\infty} p_n(\lambda)(1-\tilde{\omega})},$$

where $\tilde{c} = -\frac{\sqrt{\kappa}\sigma\gamma\theta(n-1)}{\gamma(n-1)+\frac{1}{2}+\frac{\kappa}{2}}$, and $\tilde{\omega} = \frac{\frac{\kappa}{2}}{\gamma(n-1)+\frac{1}{2}+\frac{\kappa}{2}}$. The distribution of the bids in (2.4) simplifies to $b|v \in N[c + \mu, (1-\omega)^2\kappa\sigma^2]$, which speeds up the likelihood evaluation even more.

2.3. **Gamma model.** Turning to the Gamma case, it is more convenient to solve the model in terms of *inverse* signals (Gordy, 1998). Let $s_{ij} = \frac{1}{x_{ij}}$ denotes the inverse signal of the ith bidder in auction j. Then, by following Gordy (1998), we use the following hierarchical Gamma model:

$$\begin{aligned} v_j &\sim Gamma(\xi_j, \psi_j), \;\; j = 1, ..., m, \;\; E(v_j) = \mu_j = \frac{\xi_j}{\psi_j}, \;\; Var(v_j) = \sigma_j^2 = \frac{\xi_j}{\psi_j^2} \\ s_{ij}\,|v_j &\overset{iid}{\sim} Gamma(\tau, \tau v_j), \;\; i = 1, ..., n_j, \;\; E(s_{ij}\,|v_j) = \frac{1}{v_j}, \;\; Var(s_{ij}\,|v_j) = \frac{1}{\tau v_j^2}, \end{aligned} \tag{2.8}$$

where τ works as a precision parameter, m is the total number of auctions, and n_j is the number of bidders that bid zero or place a positive bid in auction j. The bid function for the Gamma model is of the form[2]

$$b(x, \lambda) = \begin{cases} \frac{\sum_{n=2}^{\infty}(n-1)\cdot p_n(\lambda)\cdot\int_0^{\infty} v\cdot(1-F_{s|v}(1/x|v))^{n-2}\cdot f_{s|v}^2(1/x|v)\cdot f_v(v)\; dv}{\sum_{n=2}^{\infty}(n-1)\cdot p_n(\lambda)\cdot\int_0^{\infty}(1-F_{s|v}(1/x|v))^{n-2}\cdot f_{s|v}^2(1/x|v)\cdot f_v(v)\; dv}, \text{ if } x \geq x^\star \\ 0, \text{ if } x < x^\star, \end{cases} \tag{2.9}$$

[2]See Gordy (1998) for the case of a known number of bidders.

and the minimum bid for the Gamma model is given by

$$r(x^\star,\lambda)=\sum_{n=2}^{\infty}p_n(\lambda)\cdot\frac{\int_{-\infty}^{\infty}v\cdot(1-G_{s|v}(1/x^\star|v))^{n-1}\cdot g_{s|v}(1/x^\star|v)\cdot g_v(v)\;dv}{\int_{-\infty}^{\infty}(1-G_{s|v}(1/x^\star|v))^{n-1}\cdot g_{s|v}(1/x^\star|v)\cdot g_v(v)\;dv}. \tag{2.10}$$

The likelihood function changes to

$$\begin{aligned} &f_{\mathbf{b}}\left(b_2,b_3,\ldots,b_n|\mu,\sigma,\lambda,\beta,r,z\right)\\ &=\sum_{i=n}^{\bar N}p(i|\lambda)\cdot\int_{-\infty}^{\infty}\left(1-F_{s|v}\left(1/x^\star|v,\tau\right)\right)^{i-n}\cdot F_{s|v}\left[\phi\left(b_2|\beta,r,z\right)|v,\tau\right]^{I(n\geq 1)}\\ &\qquad\times\prod_{i=2}^{n}f_b\left(b_i|\beta,r,z,v\right)\cdot f_v(v|\xi,\psi)dv. \end{aligned} \tag{2.11}$$

By using the MacLaurin series of order $(\tau-1)$ for the exponential function, Gordy (1998) obtains a finite series expansion of the bid function for the Gamma model (see $B_2(x)$, formula (7)), and argues that computations are quite simple and fast. While Gordy's solution is very handy for computing the bid function, it has limited use in empirical work. First, the solution is restricted to the set of positive integers for τ, which makes it problematic if we want to link τ to covariates in the estimation process. Second, it is still very time demanding to evaluate the likelihood function since the inverse bid function has to be solved numerically for each bid in every auction. Fortunately, the equilibrium bid function for the Gamma model can be approximated in a similar way as for the approximate bid function of the Gaussian model in WV, see Appendix A for a complete derivation. The approximate bid function is given by

$$b(x)\approx\frac{[\xi+2\tau+(\lambda-2)\xi_\tau]\cdot x}{\psi x+2\tau+(\lambda-2)\psi_\tau}, \tag{2.12}$$

where

$$\xi_\tau=-0.1659-0.0159\cdot\tau+0.2495\cdot\log(1+\tau),$$

and

$$\psi_\tau=0.3095-0.0708\cdot\tau+1.0241\cdot\log(1+\tau),$$

where $0.1\leq\tau\leq 10$. If necessary, other values of τ can be handled similarly, see Appendix A for details. The minimum bid function is given by

$$b(x)\approx\frac{[\xi+2\tau+(n-1)\xi_\tau]\cdot x}{\psi x+2\tau+(n-2)\psi_\tau}. \tag{2.13}$$

Some intuition can be gained from the approximate bid function in equation (2.12). Less precision in public information (i.e. smaller ψ in the common value distribution) should constitute an incentive for a bidder to place more weight on his private information (the signal). This effect is well captured by the approximation. Replacing ξ with $\psi\mu$ in equation (2.12) and rewriting we get that

$$b(x)\approx\frac{\psi\mu+2\tau+(\lambda-2)\xi_\tau}{\psi+\frac{2\tau+(\lambda-2)\psi_\tau}{x}}, \tag{2.14}$$

and

$$b(x)\longrightarrow\frac{2\tau+(\lambda-2)\xi_\tau}{2\tau+(\lambda-2)\psi_\tau}\cdot x\;if\;\psi\longrightarrow 0,\quad\text{and}\quad b(x)\longrightarrow\mu\;if\;\psi\longrightarrow\infty.$$

Increasing the precision parameter τ (i.e. higher precision) gives an opposite effect on the approximate bid function. To see this, we first note that our approximation routine gives that

$$\left(\frac{\xi_\tau}{\tau}, \frac{\psi_\tau}{\tau}\right) \longrightarrow (0,0) \; if \; \tau \longrightarrow \infty,$$

and

$$(\xi_\tau, \psi_\tau) \longrightarrow (0,0) \; if \; \tau \longrightarrow 0.$$

It follows now directly from equation (2.14) that

$$b(x) \longrightarrow \mu \; if \; \tau \longrightarrow 0,$$

and by rewriting the approximate bid function in equation (2.12) we get that

$$b(x) \approx \frac{\left[2 + \frac{\xi}{\tau} + (\lambda - 2)\frac{\xi_\tau}{\tau}\right] \cdot x}{2 + \frac{\psi x}{\tau} + (\lambda - 2)\frac{\psi_\tau}{\tau}} \longrightarrow x \; if \; \tau \longrightarrow \infty,$$

which also agrees with intuition. The higher precision in signals the more the bidders trust their private information.

In Figure 1 the approximate bid function in equation (2.12) is compared to the exact bid function in equation (2.9). Using the estimation results in Section 3 and the eBay datasets in WV, a *representative auction*[3] is used as a benchmark. The approximation works very well for a smaller number of expected bidders, irrespective of τ and σ. As the expected number of bidders increases the approximation accuracy deteriorates. However, the approximated bids when $\lambda = 6$ are only 4.9 per cent lower than the exact bids for signals at the median of x, and only 1.9 per cent lower than the exact bids for the 99 % percentile. In addition, $\lambda = 6$ is very unrealistic according to the estimation results of the eBay auction model in Section 3.

3. Model comparison on eBay coin auction data

The performance between the Gaussian and the Gamma model is here compared using the same dataset as in WV from 1000 eBay auctions of U.S. proof coin sets.

3.1. **Posterior distribution of parameters.** In the posterior distribution the likelihood function is combined with a prior distribution on the unknown model parameters. As in WV, the user needs to typically specify four prior hyperparameters. First, the scale factor c that determines the tightness in the prior distribution of β_μ, β_σ, and β_λ. Second, the prior probability π of including a covariate in the MCMC algorithm. Third and fourth, the prior mean $\bar{\kappa}$ ($\bar{\tau}$), and the degrees of freedom g in the prior distribution of κ (τ) for the Gaussian (Gamma) model.

Table 1 reports our parameter estimates for the Gaussian and the Gamma model. With a few exceptions the estimates are fairly similar between the models. The covariates BookStd and LBookStd are the main drivers in the models for μ and σ, respectively. The large negative coefficient on MinBidStd in the Poisson entry process is due to the fact that a higher minimum bid implies a higher threshold for potential bidders to participate in the auction. In general, the posterior inclusion probabilities are either close to 0 or 1, which gives good indications of which covariates that are of importance in the models. Neither of the posterior inclusion probabilities for Book*ID, LBook*Pow, and LBook*ID are close to 1 for both models, which suggests that eBay's detailed seller information is not an obvious significant source for how

[3]The representative auction in WV is taken to be the median of the covariates and the posterior mean estimates of the model parameters, rounded to one decimal. This gives $\tau = 3, \mu = 21, \sigma = 8.4$, and $\lambda = 3.8$.

the models' expected value nor the standard deviation of the object's unknown common value are affected.

The major differences between the models are in the parameter estimates of κ, and τ, and between the parameter estimates in the models for σ. This is probably due to the differences in the model setups. The parameter σ is used in both the distributions of v and $x|v$ for the Gaussian model compared to the Gamma model with σ only used in the distribution of v. In addition, v is defined as the expected value in the distribution of $x|v$ for the Gaussian model, whereas v is used in both the expected value and the variance of the inverse signal distribution for the Gamma model. It seems that the lower value of τ, compared to κ, compensates for the generally higher values of the parameter estimates in σ for the Gamma model.

The posterior inclusion probabilities for most of the covariates are very close in magnitude between the models, especially for the covariates in λ. The major differences in inclusion probabilities for Book*ID, LBook*Pow, and LBook*ID, are due to only minor differences of the *Bayesian t-ratio*,

$$t_{Bayesian} = \frac{\left|\hat{\theta}\right|}{s(\hat{\theta})},$$

where $\hat{\theta}$ and $s(\hat{\theta})$ are the posterior mode and the approximate/asymptotic posterior deviation, respectively, from the optimization of the posterior function. WV show that the inclusion probabilities are very sensitive to changes around a certain threshold of the Bayesian t-ratios. The inclusion probabilities increase sharply above the threshold value (slightly larger than 1.96 threshold used in classical t-tests at a 5% significance level). The values of the Bayesian t-ratios, for the covariates above, differ only slightly between models but gives nevertheless substantially different inclusion probabilities. Hence, differences in inclusion probabilities which are not close to either 0 or 1, should not be treated as a sign of very different estimation results between the models.

The performance of the MCMC algorithm is better in the Gaussian case. The mean acceptance probability is 56% for the Gaussian model compared to 30% for the Gamma model, and the inefficiency factors (IF) are lower in the Gaussian case. In WV the IF is defined as the number of draws needed to obtain the equivalent result of a single independent draw. IFs close to unity is therefore an indication of a very efficient algorithm. The median, the minimum, and the maximum values of IF are $3.03, 3.80$, and 11.54 for the Gaussian model, respectively, compared to $5.08, 19.30$, and 36.80 for the Gamma model.

3.2. **In-sample fit.** Similar to BH and WV, we evaluate the in-sample fit of the models by comparing the observed data to simulated data from each model. Given a 100 systematically sampled posterior draws of the model parameters, and the observed auction specific covariates from 1000 auctions, we simulated $10,000$ full datasets with bids for each auction. Figure 2 presents within-auction bid dispersion as the difference between the highest observed bid and the lowest bid divided by the item's book value for each auction. In Figure 3, heterogeneity across auctions is presented as histograms of the bids divided by the corresponding book value in that auction. As we can see in both figures the Gaussian and the Gamma model capture both the observed within-auction dispersion and the cross-auction heterogeneity very well. The fit of the within-auction bid dispersion are nearly identical for both models, while the Gaussian model performs slightly better in fitting the cross-auction heterogeneity than the Gamma model. The histogram for the cross-auction heterogeneity of the Gamma model appears to be somewhat skewed to the left compared to the more symmetrical fit from the Gaussian counterpart.

3.3. **Out-of-sample predictions.** The differences between the models are most appearent in their out-of-sample predictions. Following WV, we use our estimated Gaussian and Gamma models to predict a dataset of 48 additional auctions of *U.S.* proof sets, which were not used in the estimation process. Given the auction specific covariates from these auctions, price distributions were simulated in a similar way as for the simulated datasets of the in-sample fits. The price distributions are a mixture between point distributions for no bids and one bid, and continuous distributions for at least two bids (when at least one bid becomes observed). The probability mass of the continuous distribution is set to equal the predictive probability of at least two bids. Hence, the continuous distribution is not normalized to 1.

The predictive price distributions for both models are displayed on top of each other for each auction in Figures 4 and 5. Auction $1-37$ contains at least two bids, $38-45$ a single bid, and $46-48$ no bids. Both models perform well in predicting auction prices with the actual price located near the center of the visible continuous price distributions. The variances of the predictive price densities for the Gaussian model appear to be smaller than for the Gamma model, where the price distributions are lower and skewed to the left.

In an attempt to quantify and summarize the visual results in Figures 4 and 5, we compute the log predictive density score (LPDS) for each auction in the evaluation dataset. Rather than trying to combine the discrete and continuous components of the predictive distribution in a more or less arbitrary way, we will compute the LPDS for each component separately.

First, to evaluate the discrete part of the distribution we compute the multinomial predictive probability of no bids, one bid, and at least two bids. Let $p_{i,j}$ be the probability of j bids in auction i, and let

$$I_{i,j} = \begin{cases} 1, & \text{if there is } j \text{ actual bids in auction } i \\ 0, & \text{if there is not } j \text{ actual bids in auction } i, \end{cases}$$

where $i = 1, \ldots, m_t$, $j = 0, 1$, and m_t is the number of test auctions used to evaluate the predictions. Let $\mathbf{I} = (I_{1,0}, I_{1,1}, I_{2,0}, \ldots, I_{m_t,1})$ be the vector of observed indicator variables, and let $\mathbf{p} = (p_{1,0}, p_{1,1}, p_{2,0}, \ldots, p_{m_t,1})$ be the predictive probabilities for each auction. Then, the discrete LPDS measure is defined as

$$LPDS_d = \frac{\sum_{i=1}^{m_t} \sum_{j=0}^{2} I_{i,j} \cdot \log p_{i,j}}{m_t}, \tag{3.1}$$

where $I_{i,2} = (1 - I_{i,0} - I_{i,1})$, and $p_{i,2} = (1 - p_{i,0} - p_{i,1})$ are the indicator variable and the predictive probability for at least two bids in auction i, respectively. Table 2 presents the $LPDS_d$ for our eBay dataset. In a majority of the 48 test auctions the Gaussian model attains a higher score than the Gamma model, even if the mean value of $LPDS_d$ is lower across all auctions for the Gaussian model. Generally, the scores from both models are much lower for the 11 auctions with maximum one bid, where the Gaussian model attains a substantially lower score than the Gamma model. In all, when it comes to the discrete part of the predictive distribution, this suggest a slight edge for the Gaussian model in auctions with at least two bids, and a slight edge for the Gamma model whenever there is maximum one bid in the auction.

The second LPDS measure focuses on the continuous part of the predictive price distribution and is defined as

$$LPDS_c = \frac{\sum_{i=1}^{m_{t_2}} \log\left(\frac{\tilde{p}(y_i)}{p_{i,2}}\right)}{m_{t_2}}, \tag{3.2}$$

where y_i is the realized price in auction i evaluated in the predictive price distribution $\tilde{p}(\cdot)$, and m_{t_2} is the number of test auctions with at least two bids. The Gaussian model outperforms the Gamma model here. As we can see in Table 2, the Gaussian model performs best in a large majority of the auctions, and the score for the Gaussian model is substantially higher compared to the Gamma model. In Figures 6 and 7 the discrepancies in scores are obvious. The Gaussian model attains, in general, higher predictive probabilities at the realized prices than the Gamma model.

The reason for the relatively good performance of the Gaussian model is probably that the true value distribution is close to normal, or at least symmetric, for this particular dataset. To infer the magnitude of the skewness in the value distribution, we obtained the values of μ and σ for the Gamma model in each of the 48 test auctions. Specifically, given the covariates in an arbitrary auction and the posterior means for the Gamma model in Table 1 the typical degree of skewness, S_k, can be calculated for each auction as

$$S_k = \frac{2}{\sqrt{\xi}} = 2\frac{\sigma}{\mu} = 2c_\nu,$$

where c_ν is the coefficient of variation. In Figure 8 a boxplot is displayed for the different degree of skewness in the 48 test auctions. In most auctions the degree of skewness is about $0.7 - 0.8$, which suggests that the value distribution is fairly symmetric compared to the results in the next section for the least skewed model. In that upcoming section we simulated data with different degrees of skewness in the value distribution to investigate the relative performance between the Gaussian and the Gamma model.

4. Model comparisons on skewed simulated data

We conducted a simulation study to compare the performance between the Gaussian and the Gamma model using positive values and different degrees of skewness, S_k, in the data-generating process. Specifically, three Gamma models with $S_k = 2$ (the exponential distribution), $S_k = 1.5$, and $S_k = 0.5$ (the least skewed model) were used to simulate 25 full datasets of bids and auction specific covariates in 1000 auctions for each model. Then, each dataset for each degree of skewness was estimated by both the Gaussian and the Gamma model.

4.1. **In-sample fit.** Since the book value does not exist as a covariate for the simulated data, we exclude the book value from the definitions of the within-auction dispersion and the cross-auction heterogeneity. Hence, the within-auction dispersion is here defined as the difference between the highest observed bid and the lowest bid, while the cross-auction heterogeneity is displayed as histograms of the bids across all auctions.

Figures 9 to 11 display the median of the within-auction dispersion and the cross-auction heterogeneity across the 25 simulated datasets for each of the three degrees of skewness. The Gamma model fits the data very well in all cases. As the degree of skewness increases the fit of both the within-auction dispersion and the cross-auction heterogeneity becomes worse for the Gaussian model. However, in Figure 11 for the least skewed data the Gamma model fits the cross-auction heterogeneity (the distribution of bids) only slightly better than the Gaussian counterpart. This is probably due to the quite symmetrical distribution of bids for the actual data in the figure. In fact, whenever the distribution of bids is symmetric the Gaussian model fits the data very well, since the bid distribution of the Gaussian model is normal for the approximate bid function in equation (2.7). In Figures $8 - 9$ the distribution of bids is highly skewed, and the Gaussian fit of the data becomes highly skewed to the right.

4.2. **Out-of-sample predictions.** As the degree of skewness increases in the data-generating process the variance in the common value distribution becomes larger. Therefore, in order to compare the $LPDS_c$ in a more appropriate way for different degrees of skewness, we adjuste the $LPDS_c$ in this case to

$$LPDS_c = \frac{\left(\sum_{i=1}^{m_{t_2}} \log\left(\frac{\tilde{p}(y_i)}{p_{i,2}}\right)\right)/m_{t_2}}{\sqrt{Var(v)}} = \frac{\psi S_k}{2}\frac{\sum_{i=1}^{m_{t_2}} \log\left(\frac{\tilde{p}(y_i)}{p_{i,2}}\right)}{m_{t_2}}, \tag{4.1}$$

where $S_k = \frac{2}{\sqrt{\xi}}$.

Table 3 reports the LPDS for each degree of skewness. The Gamma model outperforms the Gaussian model in the continuous part of the predictive price distribution, except for the lowest degree of skewness where the Gamma model is only slightly better than the Gaussian model. In average, the mean and standard deviation of $LPDS_c$ are higher and lower, respectively, for the Gamma model. However, for roughly half of the 25 datasets the Gaussian model has a higher $LPDS_c$ than the Gamma model in a majority of auctions, regardless degree of skewness. To get a closer look at possible differences, we compared the predictive price densities of the Gamma and the Gaussian model in several datasets for each degree of skewness.

In general, the densities for the Gamma model were more concentrated and skewed to the left than the Gaussian counterpart where the densities were more symmetrical and more located to the right. This resulted in substantial differences between the $LPDS_c$ of the models for realized prices to the left in the predictive distribution, but only minor differences to the right. In the left part of the predictive price densities the Gamma model was clearly the dominating model in $LPDS_c$, while the Gaussian model had only a little higher $LPDS_c$ than the Gamma model in the right part. As the degree of skewness increases these differences become larger.

Turning to the discrete part of the predictive distribution, we can discern from Table 3 that the Gamma and the Gaussian model alternate in attaining the lowest scores for at least two bids and maximum one bid. Whenever there is at least two bids the Gamma model is outperforming the Gaussian model, in average, with substantially higher scores, in contrast to the opposite effect for maximum one bid where the Gamma model attains, in average, lower scores. This is consistent with the result from the eBay dataset where the Gaussian is generally the dominating model. The mean and the precision of the scores are increasing in the degree of skewness for at least two bids, while the opposite effect pertains to the Gamma model for maximum one bid.

Moreover, the differences in scores decreases as the data becomes more symmetrical, and for the most symmetrical case the models are again equally good as for the continuous part of the predictive distribution. In fact, since the Gaussian model performs better than the Gamma model for the eBay data, and since both models seem to perform equally good for $S_k = 0.5$, there is possibly a threshold, less than $S_k = 0.5$, when the Gaussian model becomes the best fitted model for lower degrees of skewness. In all, the results from the simulated data shows that the Gamma model is clearly outperforming the Gaussian model for highly skewed data, but when the degree of skewness decreases the differences diminish.

5. Conclusions

An inherent feature of econometric auction models is that the value of the auctioned object is intrinsically non-negative. Nevertheless, there often exists value distributions in the literature that allow for negative values, even if such distributions are untenable. Our paper explores this issue by contrasting the Gaussian model in WV with an extension of the Gamma model in

Gordy (1998). A key feature in our approach is the derivation of an accurate approximation of the bid function for the Gamma model, which can be inverted and differentiated analytically. This is extremely valuable for fast and numerically stable evaluations of the likelihood function in the estimation process.

We utilized the general MCMC algorithm for Bayesian variable selection in WV to compare the relative performance between the Gaussian and the Gamma model, using WV's eBay dataset from 1000 auctions of U.S. proof coin sets. In general, the Gaussian model was slightly better than the Gamma model in fitting this data, and in predicting auction prices on the 48 auctions of the test sample. This is probably due to an almost normal or at least symmetrical value distribution, where the density is bounded away from zero with small probabilities for negative values. In fact, given the covariates for the test auctions and the posterior means for the Gamma model, we obtained small values for the coefficient of variation and the degree of skewness in most auctions, which support this explanation.

To explore possible limitations of the Gaussian model for other degrees of skewness in the value distribution, we simulated 25 datasets from three Gamma models with different degrees of skewness. Then, the relative performance between the Gaussian and the Gamma model was compared for each case. The Gamma model was clearly outperforming the Gaussian model for the two most skewed models, but was only slightly better for the least skewed model. This agrees with the result from the eBay dataset. As the value distribution becomes more symmetrical the relative performance of Gaussian model increases compared to the Gamma model. This suggest that it is of importance to make use of either model, or both, depending on the skewness of the data.

Appendix A: Approximating the equilibrium bid function for the Gamma model

The key ingredient in our approach is to approximate the survival function $(1-G_{s|v}(1/x|v))$ by a Gamma probability density function over the whole interval $(0,\infty)$. This is superior the approximation of the bid function for the Gaussian model in WV, where a certain interval needs to be obtained for the approximation of the standard normal distribution function. By substitution the distribution function of $s|v$ becomes

$$G_{s|v}(1/x|v) = \int_0^{1/x} \frac{(\tau v)^\tau}{\Gamma(\tau)} l^{\tau-1} e^{-\tau v \cdot l}\ dl = \int_0^{\tau \cdot \frac{v}{x}} \frac{1}{\Gamma(\tau)} t^{\tau-1} e^{-t}\ dt.$$

Hence, the distribution function $G_{s|v}$ depends on the parameter τ through the support $\frac{v}{x}$. Let $h\left(\frac{v}{x}|\xi_\tau,\psi_\tau\right) = \frac{\psi_\tau^{\xi_\tau+1}}{\Gamma(\xi_\tau+1)}(\frac{v}{x})^{\xi_\tau} \cdot e^{-\psi_\tau \cdot \frac{v}{x}}$ be the approximating Gamma $p.d.f.$ to $(1-G_{s|v}(1/x|v))$. Then, given an arbitrary τ the approximation constants ξ_τ and ψ_τ are obtained by minimizing the maximum divergence between $h\left(\frac{v}{x}|\xi_\tau,\psi_\tau\right)$ and $(1-G_{s|v}(1/x|v))$, i.e.

$$(\hat{\xi_\tau}, \hat{\psi_\tau}) = \min_{\xi_\tau,\psi_\tau}\left(\max_{\frac{v}{x}}\left|h\left(\frac{v}{x}|\xi_\tau,\psi_\tau\right) - \left(1-G_{s|v}(1/x|v)\right)\right|\right).$$

The approximation constants can in principal be calculated for any τ, but in practice it is more convenient to tabulate them for different values of τ. However, since the approximation of the bid function needs to be solved for any τ in the estimation process, it is even better to utilize multivariate regression, in which the dependent variables $(\hat{\xi_\tau}, \hat{\psi_\tau})$ and functions of τ as independent variables are calculated for each τ on a grid of values[4]. To receive a good fit of the regression models we choose a sufficiently wide grid for τ that contains all relevant values of τ for our datasets. Note that, if necessary, the best regression model can be estimated for any other grid that suits any other specific problem.

The regression model with the lowest adjusted R-square was chosen as the best regression model, which resulted in the following estimated regression model:

$$\hat{\xi}_\tau = -0.1659 - 0.0159 \cdot \tau + 0.2495 \cdot \log(1+\tau),$$

and

$$\hat{\psi}_\tau = 0.3095 - 0.0708 \cdot \tau + 1.0241 \cdot \log(1+\tau)$$

for $0.1 \le \tau \le 10$ with R^2 equal to 99.5% and 99.2%, respectively.

Then, by replacing $(1-G_{S|V}(1/x|v))$ with $h_v(v|\hat{\xi}_\tau,\hat{\psi}_\tau)$, the approximated bid function becomes

$$b(x) \approx \frac{\int_0^\infty v \cdot v^{(n-2)\hat{\xi}_\tau+2\tau+\xi-1} \cdot e^{-(\frac{1}{x}(\hat{\psi}_\tau(n-2)+2\tau)+\psi)\cdot v}\ dv}{\int_0^\infty v^{(n-2)\hat{\xi}_\tau+2\tau+\xi-1} \cdot e^{-(\frac{1}{x}(\hat{\psi}_\tau(n-2)+2\tau)+\psi)\cdot v}\ dv}.$$

Let $\xi_\tau' = (n-2)\hat{\xi}_\tau + 2\tau + \xi$, and let $\psi_\tau' = \frac{1}{x}(\hat{\psi}_\tau(n-2)+2\tau)+\psi$. Then, we get the approximated bid function as

$$b(x) \approx \frac{\int_{-\infty}^{\infty} Gamma(v|\xi_\tau'+1,\psi_\tau')\ dv}{\int_{-\infty}^{\infty} Gamma(v|\xi_\tau',\psi_\tau')\ dv} = \frac{\Gamma(\xi_\tau'+1)}{\psi_\tau'^{(\xi_\tau'+1)}} \cdot \frac{\psi_\tau'^{\xi_\tau'}}{\Gamma(\xi_\tau')}$$

$$= \frac{\xi_\tau'}{\psi_\tau'} = \frac{\left[\xi + 2\tau + (n-2)\hat{\xi}_\tau\right]\cdot x}{\psi x + 2\tau + (n-2)\hat{\psi}_\tau},$$

[4]Several functions of τ were used as independent variables, but we do not list all of them here.

where $Gamma(v|\xi_{\tau}^{'}, \psi_{\tau}^{'})$ denotes the p.d.f. of the Gamma distributed variable v with parameters $\xi_{\tau}^{'}$, and $\psi_{\tau}^{'}$.

References

[1] Bajari, P., and A. Hortacsu (2003), The Winner's Curse, Reserve Prices and Endogenous Entry: Empirical Insights from eBay Auctions, *The Rand Journal of Economics*, **3**(**2**), 329-355.

[2] —— (2005), Are Structural Estimates of Auction Models Reasonable? Evidence from Experimental Data, *Journal of Political Economy*, **113**(**4**), 703-741.

[3] Elyakime, B., Laffont J.J., Loisel, O., and Vuong, Q. (1997) Auctioning and Bargaining: An Econometric Study of Timber Auctions With Secret Reservation Prices, *Journal of Business & Economic Statistics*, **15**(**2**), 209-20.

[4] Gordy, B. M. (1998), Computationally Convenient Distributional Assumptions for Common-Value Auctions, *Computational Economics*, **12**, 61-78.

[5] Milgrom, P., and Weber, R. (1982), A Theory of Auctions and Competitive Bidding, *Econometrica*, **50**, 1089-1022.

[6] Paarsch, J. H. (1992), Deciding between the Common and Private Value Paradigms in Empirical Models of Auctions, *Journal of Econometrics*, **51**, 191-215.

[7] Paarsch, J. H., and Hong, H. (2006). An Introduction to the Structural Econometrics of Auction Data, Cambridge, MA: MIT Press.

[8] Wegmann, B., and Villani, M. (2008), Efficient Bayesian Inference in Structural Second Price Common Value Auctions, mimeo, Department of Statistics, Stockholm University, and Sveriges Riksbank.

Table 1. Comparing the estimation results from the Gaussian and the Gamma model on the eBay dataset.

Parameter	Covariate	Post Mean		Post Std		Post Incl prob	
		Gaussian	Gamma	Gaussian	Gamma	Gaussian	Gamma
κ/τ	-	5.499	2.997	0.772	0.111	1.000	1.000
μ	Const	28.273	28.307	0.245	0.304	1.000	1.000
	BookStd	0.740	0.747	0.010	0.012	1.000	1.000
	Book*Pow	0.033	0.046	0.015	0.018	0.064	0.107
	Book*ID	0.128	0.052	0.036	0.039	0.900	0.017
	Book*Unopen	0.372	0.488	0.029	0.051	1.000	1.000
	Book*MinBlem	−0.022	0.002	0.021	0.028	0.010	0.008
	Book*MajBlem	−0.252	−0.269	0.030	0.040	1.000	1.000
	Book*LargNeg	−0.003	−0.020	0.018	0.025	0.004	0.009
$\log(\sigma^2)$	Const	3.997	4.314	0.071	0.038	1.000	1.000
	LogBookStd	1.262	1.276	0.038	0.026	1.000	1.000
	LBook*Pow	0.043	0.069	0.018	0.020	0.220	1.000
	LBook*ID	0.042	0.032	0.040	0.067	0.481	0.011
	LBook*Unopen	0.211	0.362	0.027	0.019	1.000	1.000
	LBook*MinBlem	−0.028	−0.057	0.027	0.026	0.012	0.039
	LBook*MajBlem	0.036	0.063	0.040	0.049	0.007	0.017
	LBook*LargNeg	0.035	0.042	0.021	0.027	0.017	0.050
$\log(\lambda)$	Const	1.193	1.234	0.021	0.022	1.000	1.000
	Pow	0.009	−0.028	0.035	0.029	0.005	0.012
	ID	−0.177	−0.197	0.110	0.078	0.030	0.048
	Unopen	0.323	0.331	0.048	0.048	1.000	1.000
	MinBlem	−0.049	−0.042	0.048	0.048	0.008	0.009
	MajBlem	−0.151	−0.115	0.085	0.097	0.019	0.015
	LargNeg	0.055	0.086	0.049	0.047	0.012	0.022
	LogBookStd	−0.038	−0.036	0.027	0.021	0.018	0.031
	MinBidStd	−1.433	−1.380	0.056	0.059	1.000	1.000

Note: The prior hyperparameters are set to $c = n$, $\bar{\tau} = 0.25$, $g = 4$, and $\pi = 0.2$, as for the benchmark model in WV.

Table 2. Log predictive density scores (LPDS) on the eBay coin auction data for the Gaussian and the Gamma model.

Test auction	Measure	Gaussian	Gamma	% Gaussian wins
$1-37$	$LPDS_c$	-3.547	-3.684	27/37
$1-48$	$LPDS_d$	-0.526	-0.515	25/47
$1-37$	$LPDS_d$	-0.212	-0.238	20.5/37
$38-48$	$LPDS_d$	-1.687	-1.539	5/11

Note: Auctions 1-37 contains at least two bids, and auctions 38-48 maximum one bid. $LPDS_d$ for auction number 43 is undefined ($-\infty$) for the Gamma model, and is therefore excluded from the calculations of $LPDS_d$. The Gaussian and the Gamma model are equally good in $LPDS_d$ for auction 21.

Table 3. Log price density scores (LPDS) on simulated data for the Gaussian and the Gamma model.

Measure	Statistics	$S_k = 2$		$S_k = 1.5$		$S_k = 0.5$	
		Gaussian	Gamma	Gaussian	Gamma	Gaussian	Gamma
$LPDS_c$	Mean	−2.430	−2.138	−2.814	−2.361	−3.173	−3.157
	Std Dev	0.254	0.186	0.833	0.166	0.109	0.095
$LPDS_d$	Mean	−0.886	−0.877	−0.764	−0.755	−0.363	−0.364
	Std Dev	0.069	0.081	0.070	0.084	0.092	0.093
$LPDS_d$	Mean	−0.584	−0.449	−0.383	−0.301	−0.111	−0.111
$Bids \geq 2$	Std Dev	0.057	0.042	0.035	0.031	0.011	0.011
$LPDS_d$	Mean	−1.372	−1.567	−1.742	−1.924	−2.612	−2.630
$Bids \leq 1$	Std Dev	0.120	0.126	0.103	0.127	0.244	0.264

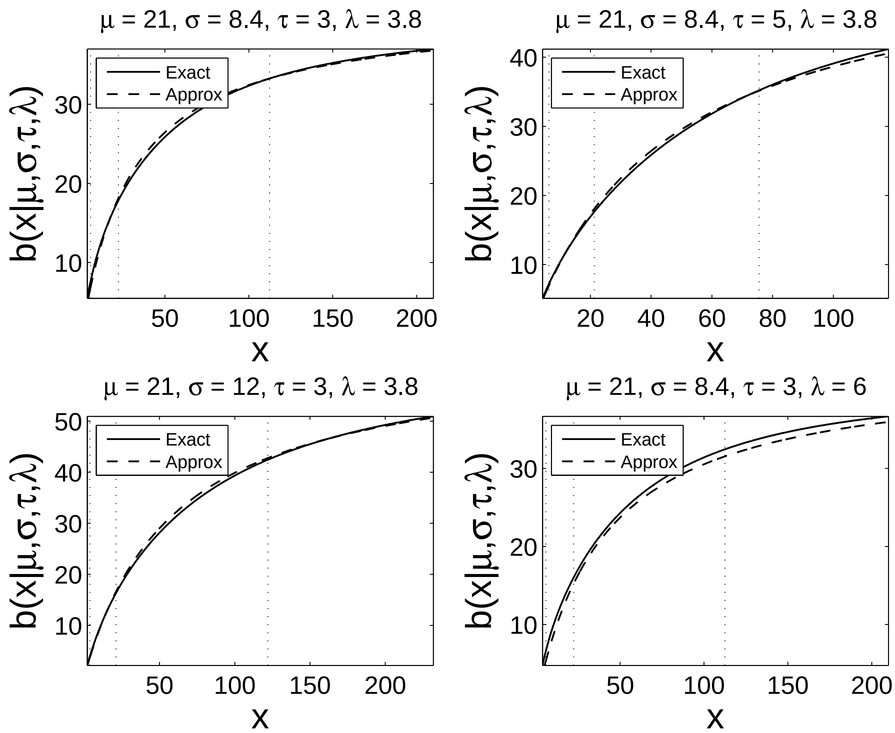

FIGURE 1. The exact versus the approximate bid function for the Gamma model. Dotted lines represent the median and the 95 % probability intervals in the unconditional distribution of x, and the full interval on the x-axis serves as 99 % probability intervals. In the upper left part of the figure the representative auction, described in Section 2.3, is used as a benchmark.

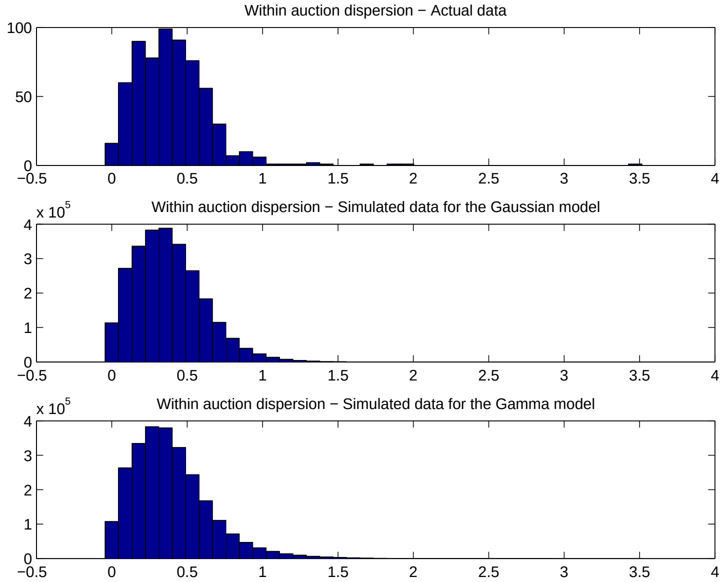

FIGURE 2. Posterior predictive comparison of the within-auction dispersion for the Gaussian and the Gamma model. The within-auction dispersion is defined as the difference between the highest observed bid and the lowest bid divided by the auctioned item's book value.

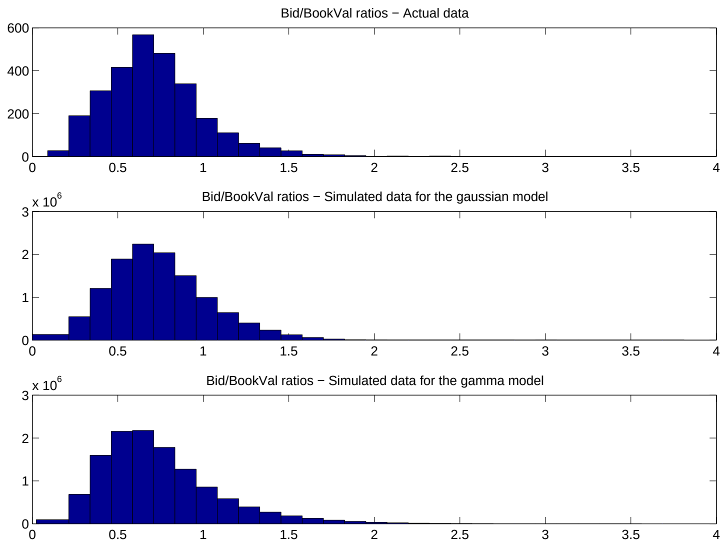

FIGURE 3. Posterior predictive comparison of the cross-auction heterogeneity for the Gaussian and the Gamma model. Cross-auction heterogeneity is displayed as histograms of the bids divided by the corresponding book value in that auction.

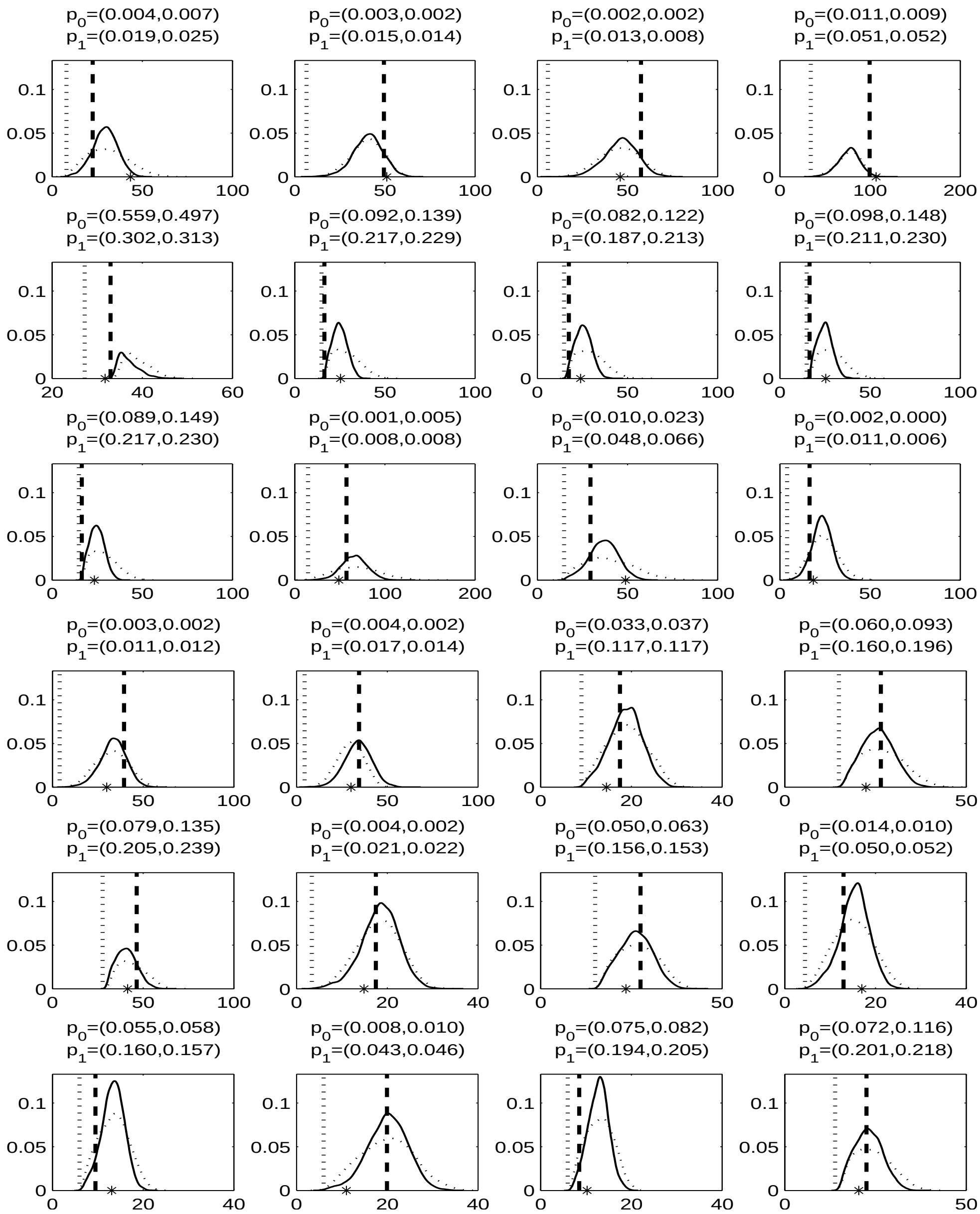

Figure 4. Out-of-sample predictions for auctions 1-24 in the evaluation sample. Each subplot displays the realized price marked out by a star (if the item is sold), the minimum bid (vertical dotted line), and the book value (vertical dashed line). The title of each subplot displays the probability of zero bids (p_0) and exactly one bid (p_1), where the first value in parenthesis comes from the Gaussian model and the second value from the Gamma model. The solid and dotted curves represent the predictive price densities for the Gaussian and the Gamma model, respectively, conditional on at least two bids.

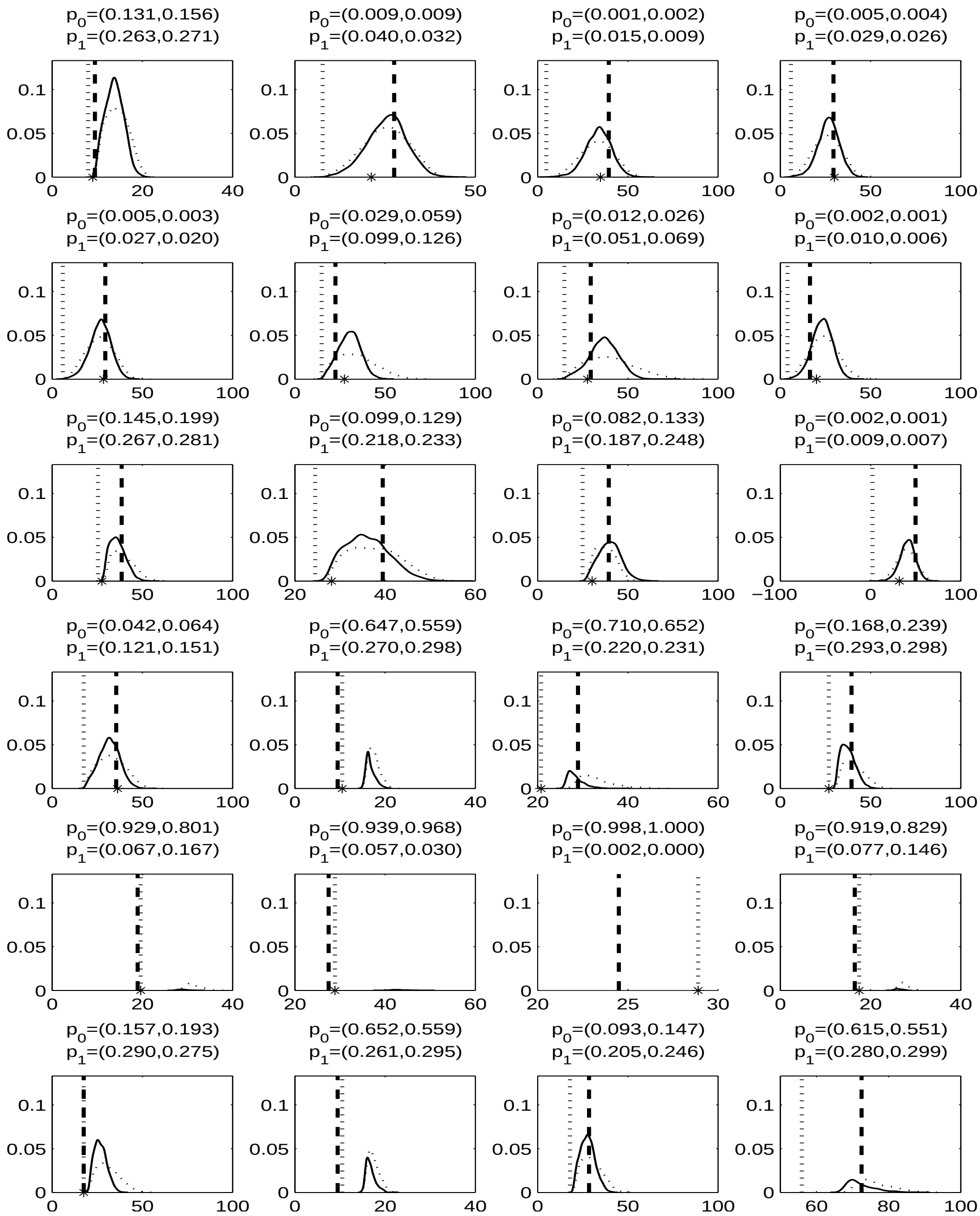

FIGURE 5. Out-of-sample predictions for auctions 25-48 in the evaluation sample. Each subplot displays the realized price marked out by a star (if the item is sold), the minimum bid (vertical dotted line), and the book value (vertical dashed line). The title of each subplot displays the probability of zero bids (p_0) and exactly one bid (p_1), where the first value in parenthesis comes from the Gaussian model and the second value from the Gamma model. The solid and dotted curves represent the predictive price densities for the Gaussian and the Gamma model, respectively, conditional on at least two bids.

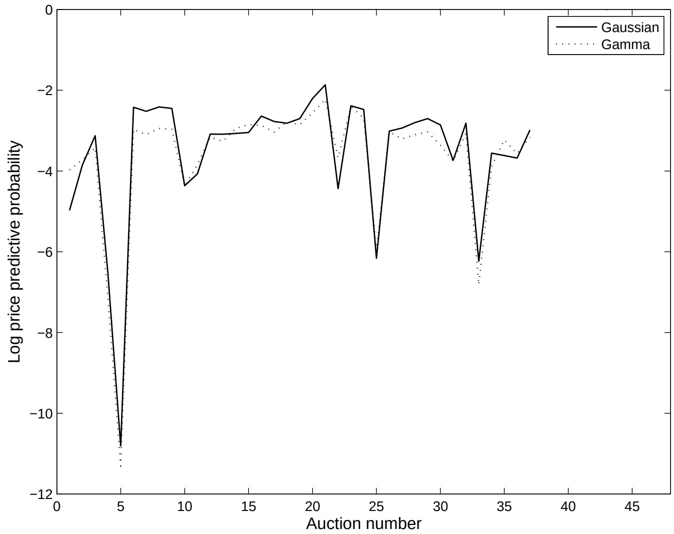

FIGURE 6. Log price predictive probabilities evaluated at the actual price in each auction, conditional on at least two bids for the Gaussian and the Gamma model. Auctions 38-45 contains a single bid, and auctions 46-48 no bids.

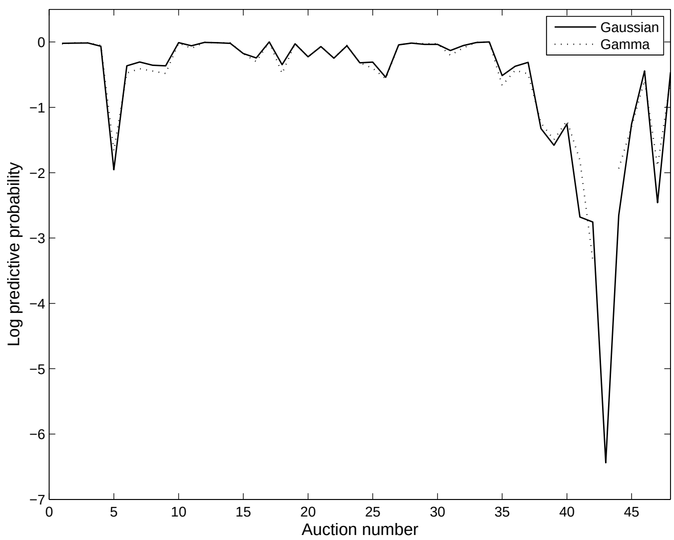

FIGURE 7. Log predictive probabilities for the actual outcome of either no bids, one bid, or at least two bids in each auction for the Gaussian and the Gamma model. Auctions 1-37 contains at least two bids, auctions 38-45 a single bid, and auctions 46-48 no bids. The log predictive probability in auction 43 for the Gamma model is not displayed since the probability equals zero.

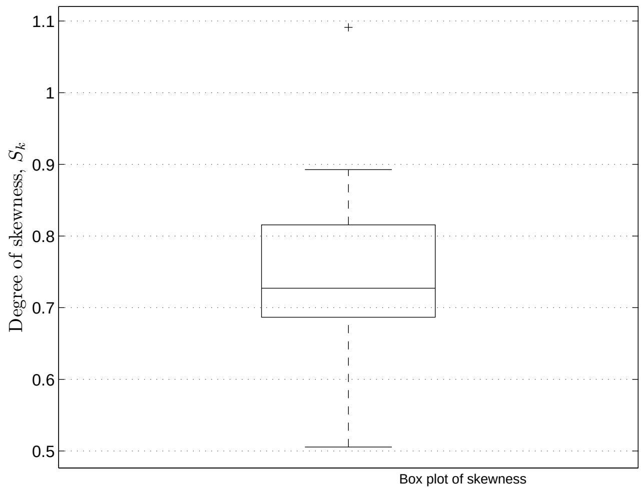

FIGURE 8. Box plot for the estimated skewness in the 48 test auctions given the auction specific covariates, and the posterior means for the Gamma model in Table 1.

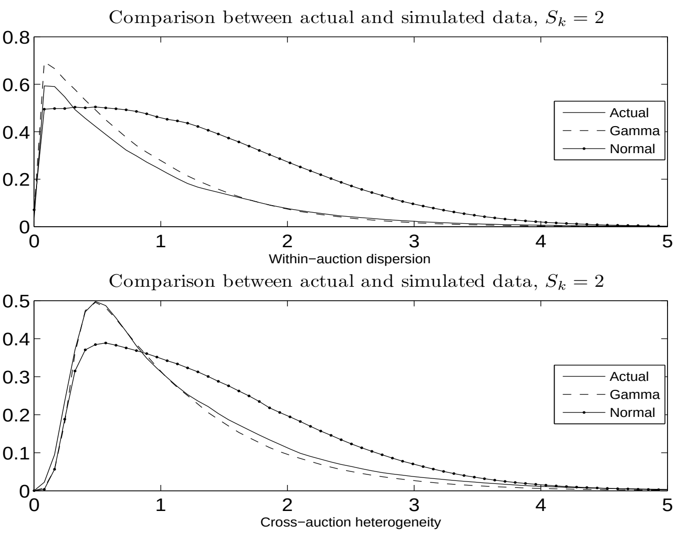

FIGURE 9. Posterior predictive comparisons of the within-auction dispersion and the cross-auction heterogeneity for the Gamma and the Gaussian model. The within-auction dispersion is defined as the difference between the highest observed bid and the lowest bid. Cross-auction heterogeneity is displayed as histograms of the bids across all auctions. The degree of skewness for the simulated data, S_k, is equal to 2.

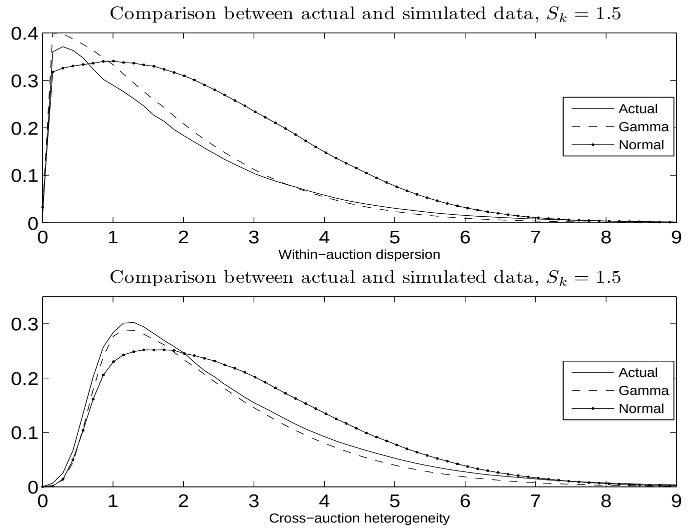

Figure 10. Posterior predictive comparisons of the within-auction dispersion and the cross-auction heterogeneity for the Gamma and the Gaussian model. The within-auction dispersion is defined as the difference between the highest observed bid and the lowest bid. Cross-auction heterogeneity is displayed as histograms of the bids across all auctions. The degree of skewness for the simulated data, S_k, is equal to 1.5.

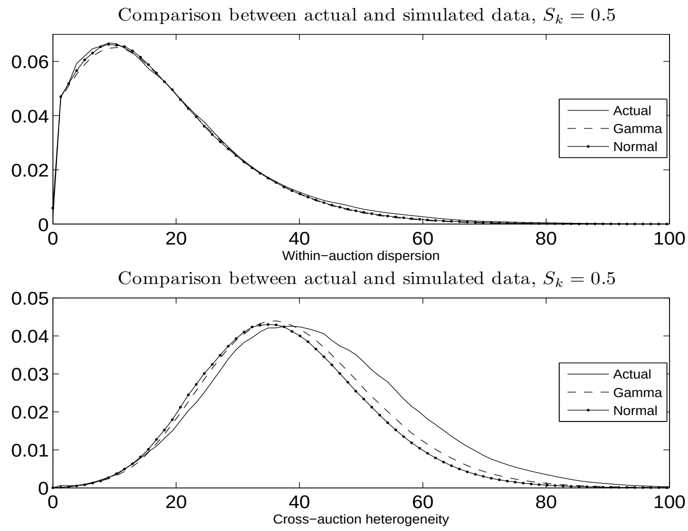

Figure 11. Posterior predictive comparisons of the within-auction dispersion and the cross-auction heterogeneity for the Gamma and the Gaussian model. The within-auction dispersion is defined as the difference between the highest observed bid and the lowest bid. Cross-auction heterogeneity is displayed as histograms of the bids across all auctions. The degree of skewness for the simulated data, S_k, is equal to 0.5.